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Barycentric coordinates for polytopes

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ABSTRACT

In Wachspress (1975) [1], theory was developed for constructing rational basis functions for convex polygons and polyhedra. These barycentric coordinates were positive within the elements. Generalization to higher space dimensions is described here. The GADJ algorithm developed by Dasgupta (2003) [5] and in Dasgupta and Wachspress (2008) [6] is crucial for simple construction of rational barycentric basis functions.

1. Polytopes

A d-dimensional space has a coordinate basis $x_k,\ k = 0, 1, 2, \ldots, d$. The homogeneous coordinate x_0 is unity in affine coordinates and zero on the absolute hyperplane (horizon). A d-plane is a linear combination of these coordinates: $\sum_0^d (a_k x_k)$, where the a_k are real rational coefficients. A polytope is a simple region bounded by d-planes. It is convex if any line segment connecting two points in the region lies in the region. A vertex is simple if the number of d-planes meeting at the vertex is equal to d. A polytope is simple if all of its vertices are simple. All vertices in two space are simple. In early work (Wachspress [1]), only simple vertices were allowed in three dimensions. More recently, this was extended to vertices of any order by Warren [2] and Wachspress [3]. Higher-order boundary components have been analyzed in two and three dimensions. A polytope is bounded by faces of dimension $d-1$. Vertices (of dimension zero) of a simple polytope are points common to d faces. Edges (dimension 1) connect adjacent vertices.

2. Barycentric coordinates

Barycentric coordinates satisfy certain conditions. A barycentric coordinate (basis function) is associated with each vertex of a polytope. These functions sum to unity. Each coordinate varies linearly on edges that meet at the vertex and vanishes on all opposite faces. The barycentric coordinates form a basis for all space coordinates. Each barycentric coordinate is nonnegative over the element and positive interior to the element. Each coordinate is continuous over the element. Warren et al. [4] established the existence of rational barycentric coordinates over convex polytopes of any dimension. They asserted that “Unfortunately, no explicit formulation of these functions has been provided for practical implementation”. In this note, an explicit formulation will be described.

3. The GADJ algorithm

The GADJ algorithm was introduced by Dasgupta [5] for convex polygons and extended in [6] to elements with curved sides in two and three dimensions. The algorithm for convex polyhedra provides a basis for extension to higher-dimension spaces. Barycentric coordinates (one for each vertex) sum to unity. If all are rational with a common denominator, then the

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sum of the numerators is equal to the denominator. If a polyhedron is bounded by n planes, a unique surface of maximal order $n-4$ on which the denominator vanishes may be computed from the divisor group (excluding the vertices) of the boundary planes. The denominator is found easily by the GADJ algorithm. Let vertices j and k have the common edge (j,k). The planes intersecting on (j,k) are F_j and F_k. The third plane at vertex j is P_j and at k is P_k. The other boundary planes common to vertices j and k are R_{jk}. Let the numerator of the barycentric coordinate at j be s_jN_j, where $s_1 = 1$ and the remaining s_j are to be determined. $N_j = s_jR_{jk}P_k$ and $N_k = s_kR_{jk}P_j$. These are the only coordinates which contribute to the value of the sum of the coordinates on side (j,k). Each coordinate is linear on side (j,k). It follows that

$$\frac{s_jN_j}{s_jN_j + s_kN_k} = \frac{s_jP_k}{s_jP_k + s_kP_j}$$

is linear on (j,k). The numerator is linear. Hence, the denominator is constant on (j,k). Since P_k is zero at k and P_j is zero at j, it follows that

$$s_k = \frac{P_k \text{ at } j}{P_j \text{ at } k} s_j. \tag{1}$$

Uniqueness of the denominator assures that the s_k are independent of the order in which they are computed. Any cycle back to vertex 1 will yield $s_1 = 1$. This may be used to verify the accuracy of the recursive algorithm. Now the denominator polynomial is simply

$$Q_{n-4} = \sum_1^n (s_jN_j). \tag{2}$$

When the element is convex, all face polynomials may be normalized to be positive within the element. Then all the s_k will be positive, and both Q and the barycentric coordinates will be positive within the element. The algorithm fails when the element is not convex in that Q vanishes at points within the element.

4. Higher-dimension polytopes

The generalization to d dimensions is apparent. A simple vertex in d-space is of order d. Let the polytope be bounded by n d-planes. The numerator at j is $s_jR_{jk}P_k$. Edge (j,k) is at the intersection of $d-1$ d-planes common to vertices j and k. The remaining plane meeting at j is P_j. Similarly, P_k is the remaining plane at k. Now R_{jk}, common to the numerators at j and k, is of order $n-d-1$. The variation of the barycentric coordinate associated with vertex j on edge (j,k) is

$$\frac{s_jP_k}{s_jP_k + s_kP_j},$$

and the recursive algorithm for the s_k is again given by Eq. (1). The denominator is found by Eq. (2). If a rational barycentric basis exists, then this algorithm may be used to generate the basis. That the algorithm produces coefficients independent of the order in which they are generated may be readily established by considering a consistent block ordering in which sets are generated by next-nearest-neighbor ordering. Since Warren et al. [4], have established the existence of a rational barycentric basis, this algorithm establishes uniqueness.

Vertices that are not simple require more delicate analysis. Following Warren's treatment, we may truncate the element with a d-plane near a vertex of order r to obtain r simple vertices and let this plane approach the vertex. The construction given in [3] is based on this procedure but eliminates the limiting process. An "adjacent" factor S_j of order $r-d$ is introduced at vertex j of order $r > d$. This factor is unity at a simple vertex.

The recursion allowing for adjacent factors is

$$s_k = \frac{P_kS_k \text{ at } j}{P_jS_j \text{ at } k} s_j. \tag{3}$$

The GADJ algorithm generalizes to faces of higher order. The analysis establishing uniqueness of a normalized denominator constructed from the divisor of the element boundary polynomial segments is not difficult in two dimensions, but it becomes quite complicated in three dimensions even when the faces are restricted to planes or quadrics. It seems plausible that unique coordinates exist when the bounding faces are rational algebraic surfaces in the d dimensions. However, analysis appears to be extremely complicated.

5. A 4D example

The tesseract is a 4D polytope with Schafli symbol (4, 3, 3) [7]. It has 16 vertices, 32 edges, 24 "bodies", and 8 "hyperplane" faces. It is simple in that each vertex is of order 4. Four hyperplanes meet at each vertex, three of which are common to j and k along edge (j,k), which is just the intersection of these three hyperplanes in 4-space. The fourth hyperplanes at j and k appear in the generation of s_k from s_j. We rely on the Warren et al. proof of existence of a barycentric coordinate basis. However, it is instructive to try to extend the 3D proof in [1]. The eight hyperplane faces meet in ${}_8C_4 = 70$ points of multiplicity 4. 16 of these are vertices. This yields 54 exterior 4-points through which we wish to pass a 4D surface of order $n - 5 = 3$. (Note that the denominator in 2D is order $n - 3$, in 3D is $n - 4$, and in 4D is $n - 5$. In algebraic geometry nomenclature this is a "special adjoint" of the boundary surface if we exclude vertices from the points in this surface.) Each exterior edge is included in this surface. The boundary faces meet in ${}_8C_3 = 56$ edges, 32 of which are element edges. Thus there are 24 exterior edges. An edge lies within a surface of order 3 when four points are in the surface. There are five hyperplane faces other than the three which form the edge. These hyperplanes intersect at five 4-points on the edge, only four of which are needed to place the edge in the surface. This is the 4D generalization of Desargues' theorem. We may subtract one point for each of these edges to obtain a total of $54 - 24 = 30$ exterior 4-points through which the denominator surface must pass.

The number of degrees of freedom in a 4D surface of order $m - 5 = 3$ is ${}_7C_4 - 1 = 34$. There are four extra degrees of freedom. But a unique rational basis exists. Where are the four additional points? The 3D divisor construction does not seem to extend to 4D. Further study is required. A detailed analysis of the denominator constructed by the GADJ algorithm is needed. The 30 exterior 4-points will certainly lie in the denominator surface. It would be of great interest to ascertain whether the other exterior 4-points can be identified from the element boundary. Other 4D elements must be analyzed. Although establishing existence of a unique denominator from the boundary divisors seems overly ambitious, the GADJ algorithm does yield the barycentric coordinates.

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